Gluttonous: A Greedy Algorithm for Steiner Forest

Vihang Agarwal, Aditya Modi, Shun Zhang

University of Michigan

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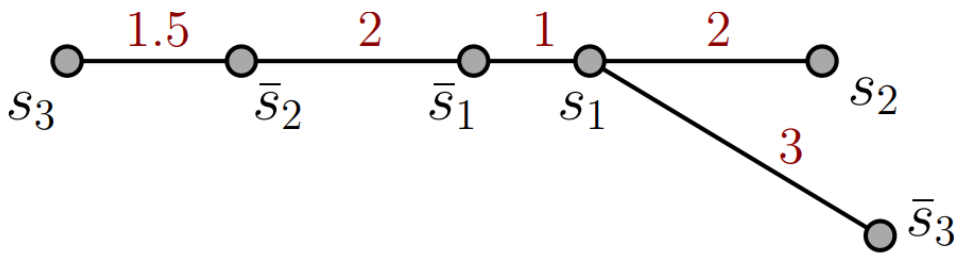
The Steiner Forest Problem

- $I = (M, D)$ is a Steiner forest instance.
  - A metric space $M = (V, d)$: V is a set of vertices; $d(\cdot, \cdot)$ is a metric.
  - Terminals $D \subseteq \binom{V}{2}$: a collection of pairs of vertices (for example, $D = \{\{s_1, \bar{s}_1\}, \{s_2, \bar{s}_2\}, \dots\}$).
- A solution F to the Steiner forest instance I is a forest: for any $\{s, \bar{s}\} \in D$, there exists a tree $T \in F$ such that $\{s, \bar{s}\} \subseteq T$.
- An optimal solution minimizes $\mathtt{cost}(F) = \sum_{T \in F} \mathtt{cost}(T)$.

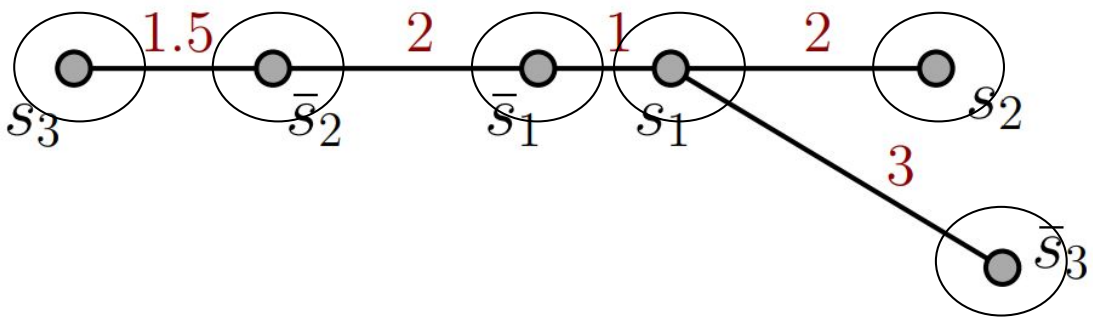


Previous Work

1. A primal-dual method (Agrawal et al., 1995; Goemans and Williamson, 1995) provides a 2-approximation algorithm.
2. Greedy algorithms.
    1. A *paired* greedy algorithm (Chen et al., 2010) is a $\Omega(\log n)$-approximation.
    2. This work (Gupta and Kumar, 2015) provides a constant-factor approximation algorithm by *ignoring the pairing relation*.

The Gluttonous Algorithm

1: initialize $C^{(0)}$ to be the trivial clustering; the set of added edges $E' \leftarrow \emptyset$; iteration index $i \leftarrow 0$
2: **while** there exist active supernodes in $C^{(i)}$ **do**
3: find active supernodes $S_1, S_2 \in C^{(i)}$ with the minimum $C^{(i)}$-punctured distance.
4: $C^{(i+1)} \leftarrow C^{(i)} \setminus \{S_1, S_2\} \cup \{S_1 \cup S_2\}$.
5: find the shortest path from any $u \in S_1$ and $v \in S_2$ under metric $M \setminus C^{(i)}$; add to E' the edges on this path
6: $i \leftarrow i + 1$
7: **end while**
8: return a maximal acyclic subgraph of E'



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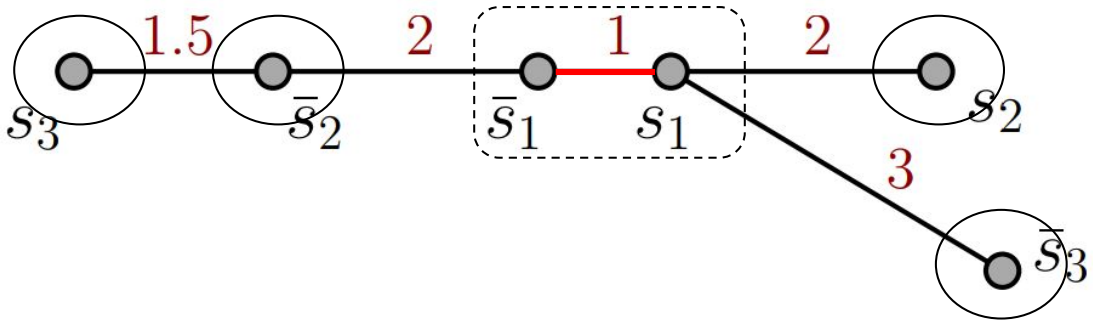

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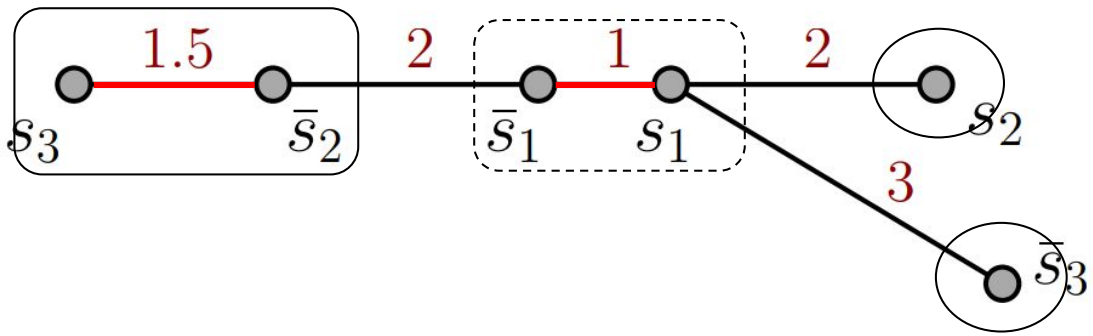

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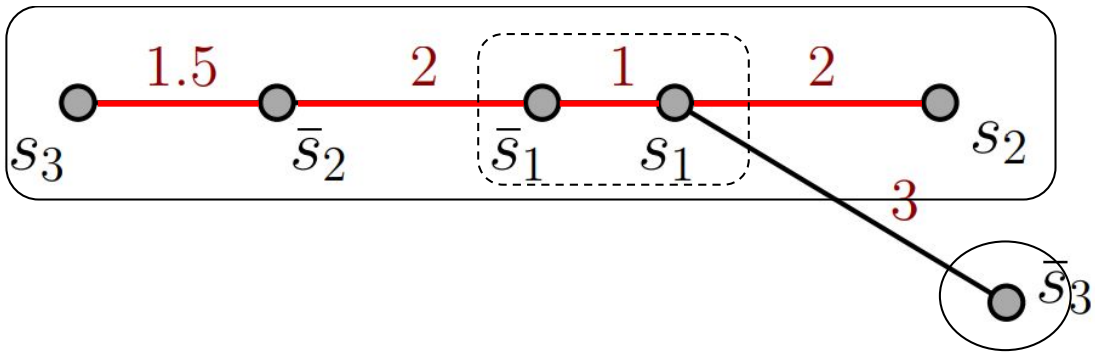

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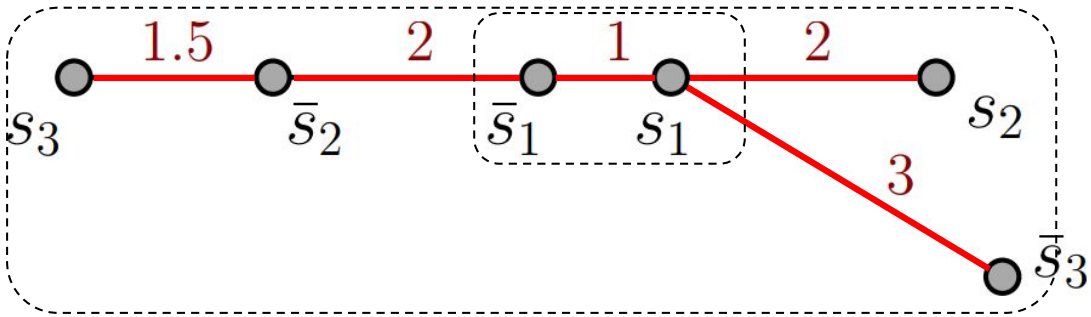


Analysis of Gluttonous

The main result:

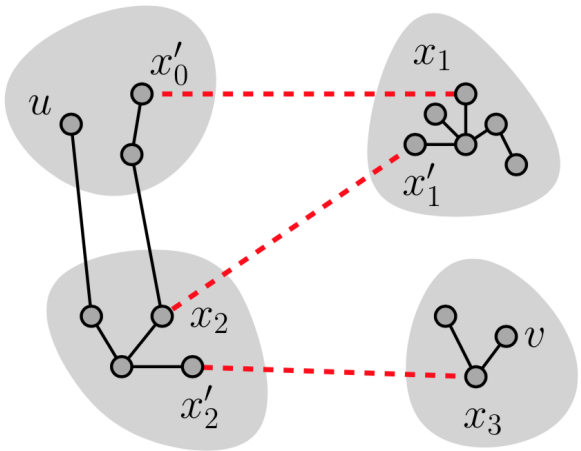
Theorem

The gluttonous algorithm is a constant-factor approximation for Steiner Forest.

For analysis, we first define *faithfulness*.

Definition

A forest F is **faithful** to a clustering C if each supernode $S \in C$ (all vertices in S) is (are) contained within a single tree in F.



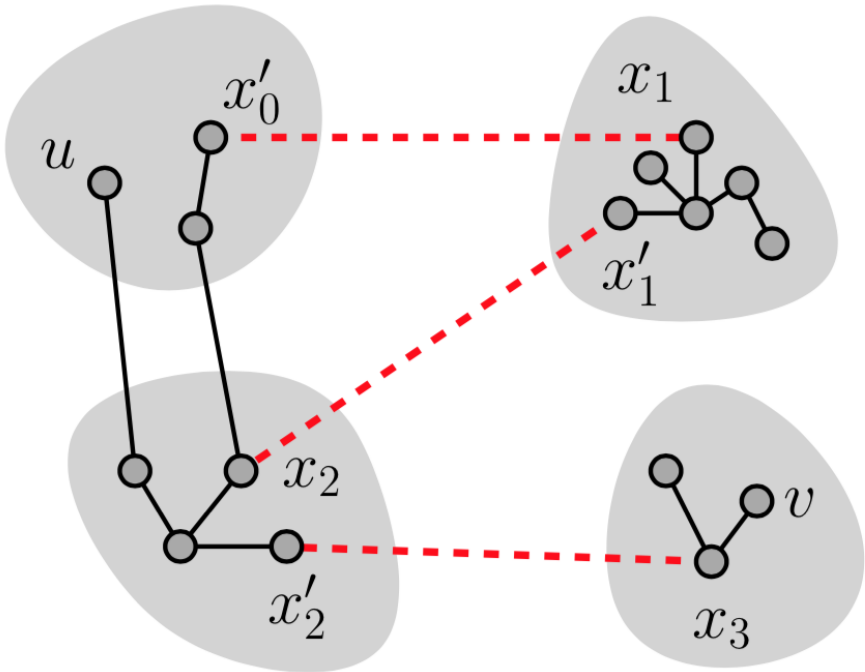
Analysis of Gluttonous

The analysis proves the bound in the following two steps:

- There exists a solution F that is faithful to C^g and $\texttt{cost}(F) \leq 2\texttt{cost}(F^*)$.
- For any solution F that is faithful to C^g, $\texttt{cost}(F^g) \leq O(1) \cdot \texttt{cost}(F)$.

Bounding the cost of F

- Optimal forest solution is F^*.
- Gluttonous returns a forest F^g with a clustering structure.
- **Idea:** Build the forest F by modifying F^* with two properties.
  1. F is faithful to the clustering of *Gluttonous* F^g.
  2. $\texttt{cost}(F) \leq 2\texttt{cost}(F^*)$.



Bounding the cost of F

Fact: F^* is optimal but most likely not faithful.
Question: How to ensure faithfulness?
Idea: Inductively build F using F^* which is faithful to *Gluttonous*.

Assume current F is faithful to current clustering.
If we connect two unmatched nodes u and v in S and S':

1. Both are in same tree in F. Unchanged F is still faithful!
2. Both lie in different T_1 and T_2. Bound cost of used path P as:

$$\texttt{cost}(P) \leq \min(d_{T_1}(u, \bar{u}), d_{T_2}(v, \bar{v})) \leq \min(\texttt{width}(T_1), \texttt{width}(T_2))$$

Next: $\texttt{cost}(F) =$ sum of all path costs $\texttt{cost}(P)$.

Theorem (Low-cost and Faithful F)

Sum of all path costs and, therefore, F is $cost(F^*) + \sum_{T' \in F^*} width(T') \leq 2cost(F^*)$.

Charging the cost of Gluttonous

Result: There exists forest F faithful to $\mathcal{C}^g$ with $\texttt{cost}(F) \leq 2\texttt{cost}(F^*)$.
Question: How to relate the cost of F to $\texttt{cost}(F^g)$?
Approach: For faithful $\bar{F}$, charge total $\texttt{cost}(F^g)$ as $\mathcal{O}(1) \cdot \texttt{cost}(\bar{F})$.

Consider any feasible solution $\bar{F}$. Now,

- For any iteration t, build a Steiner forest instance $\mathcal{I}_t$ on supernodes.
- We also maintain a forest F_t such that:
  1. F_t is feasible to the instance I_t.
  2. F_t maintains the connectivity of $\bar{F}$:
     u, v in same tree in $\bar{F} \Rightarrow S_u$, S_v in same tree in F_t

Approach:

- Start with individual nodes as clustering.
- Set the initial forest solution to $\bar{F}$
- At any step, when merging two supernodes: merge nodes in F_t (*delete* edges to remove cycle)
- To maintain cost, short cut low degree Steiner vertices (inactive supernodes).

Final result

Theorem

Given any forest $\bar{F}$, we can charge the cost of gluttonous to at most the cost of $48 \cdot \texttt{cost}(\bar{F})$.

- Cost of *bought* paths can be charged to the deleted edges.

Theorem (Approximation factor of Gluttonous)

Combining the two results, we can show that the approximation factor of Gluttonous is 96 (constant approximation).

- Choose the faithful clustering from first part as $\bar{F}$.
- $\texttt{cost}(\bar{F}) \leq 2 \cdot \texttt{cost}(F^*)$.
- $\texttt{cost}(F^g) \leq 48 \cdot \texttt{cost}(\bar{F})$.
- **Note** that the intermediate forest $\bar{F}$ is only considered in analysis.

Cost Sharing Mechanisms

Intuition: Informally, a cost-sharing mechanism builds a network connecting agents to their desired source, and allocates the cost incurred among the agents, s.t.

- No group of agents is charged too much precluding the possibility of their being unhappy and trying to leave the system.

Cost Shares for Steiner Forest

- **Definition:** Cost sharing method χ for the Steiner Forest game. (I is the Steiner Forest instance).
  - Seen as a charging function which divides the cost of connecting terminal pairs in I amongst the agents that wish to establish this connection.
- **Constraints,**
  - Sum of such cost shares should at least be equal to the total cost of the connections. i.e., $\sum_{(u,\bar{u})\in D} \chi(I,(u,\bar{u})) \leq \alpha \cdot cost(F^*)$. ($\alpha$-approximate budget-balanced).
  - Cost share of any fixed individual agent should never decrease as other agents leave this system.(cross-monotonic)

Strict Cost Shares

- χ is defined using the timed version of gluttonous algorithm. This is γ-approximation algorithm.
  - The timed version divides execution into stages i where all supernodes with merging distance in $[2^i, 2^{i+1}]$ are merged instead of the nearest active supernode pair.
- **Getting cost shares:**
  - At each stage increment the cost-share of $(u, \bar{u})$ and $(u', \bar{u}')$ by $\frac{2^{i+1}}{2\gamma}$ (u and u' are active terminals with maximum distance to their mates for a pair of supernodes respectively).

Remarks

- This cost sharing method and its strictness property ensures for an algorithm A, if we partition D arbitrarily into $D_1 \cup D_2$, and build a solution $A(D_1)$, then the cost-shares of the terminals in D_2 would suffice to augment the solution $A(D_1)$ to one for D_2 as well.
- Showing strict cost shares for Steiner forest had remained an open problem, and this paper provides a constant factor approximation strict cost share scheme using the structure and analysis for the gluttonous algorithm.

Future Direction

- **Better approximation factor:**
  - 96 and 69 constant factor obtained for greedy and local search algorithms respectively. Best approximation factor available is 2 - 1/k.
  - Thus, the problem of refining the bound and simplifying the analysis is wide open.
- **Dynamic Steiner forest**
  - Here, terminal pairs arrive online and we want to maintain a constant-approximate Steiner Forest while changing the solution by only a few edges in each update.
  - Dynamic Steiner tree algorithms have been based on local search. Thus obtaining such a solution for this generalized version through local search approximation algorithms is another possibility.
- **Stochastic multi-stage Steiner forest**
  - In the stochastic version we are given a distribution over demands and demand set is revealed in the future.
  - The idea is to use cost-sharing schemes to minimize the total expected cost. It would be interesting to show better approximation for these problems. (primal-dual methods give an approximation factor of 5).

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